Graph unique-maximum and conflict-free colorings

Panagiotis Cheilaris
Center for Advanced Studies in Mathematics
Ben-Gurion University
panagiot@math.bgu.ac.il

Géza Tóth*
Rényi Institute
Hungarian Academy of Sciences
geza@renyi.hu

Abstract

We investigate the relationship between two kinds of vertex colorings of graphs: unique-maximum colorings and conflict-free colorings. In a unique-maximum coloring, the colors are ordered, and in every path of the graph the maximum color appears only once. In a conflict-free coloring, in every path of the graph there is a color that appears only once. We also study computational complexity aspects of conflict-free colorings and prove a completeness result. Finally, we improve lower bounds for those chromatic numbers of the grid graph.

Keywords: unique-maximum coloring, ordered coloring, vertex ranking, conflict-free coloring

1 Introduction

In this paper we study two types of vertex colorings of graphs, both related to paths. The first one is the following:

Definition 1.1. A *unique-maximum coloring with respect to paths* of $G = (V, E)$ with k colors is a function $C : V \to \{1, \dots, k\}$ such that for each path p in G the maximum color occurs exactly once on the vertices of p. The minimum k for which a graph G has a unique-maximum coloring with k colors is called the *unique-maximum chromatic number* of G and is denoted by $\chi_{\mathrm{um}}(G)$.

Unique maximum colorings are known alternatively in the literature as *ordered colorings* or *vertex rankings*. The problem of computing unique-maximum colorings is a well-known and widely studied problem (see e.g. [11]) with many applications including *VLSI design* [12] and *parallel Cholesky decomposition* of matrices [13]. The Cholesky decomposition method is used in solving sparse linear systems $Ax = b$, whenever A is a symmetric $n \times n$ positive-definite matrix, and is faster than the more general LU decomposition. In [13], given a symmetric $n \times n$ positive-definite matrix A, a graph $G(A)$ on n vertices is defined which encodes the data dependencies between different columns in the linear system. The unique-maximum chromatic number of $G(A)$ is a rough estimate of the work required in parallel Cholesky decomposition of matrix A. The problem is also interesting for the Operations Research community, because it has applications in *planning efficient assembly of products* in manufacturing systems [10]. In general, it seems that the vertex ranking problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.) Another application of unique-maximum colorings is in estimating the worst-case complexity of *finding local optima in*

*Supported by OTKA T 038397 and 046246.

neighborhood structures. A neighborhood structure is a connected graph in which every vertex has a real value. Suppose that we want to find a vertex v which is a local optimum. For example, if v is a local minimum, then its value is not greater than the values of its adjacent vertices. The goal is to query as few vertices of the neighborhood structure as possible. In some classes of bounded-degree neighborhood structures (like grids), the worst-case complexity of finding a local optimum is closely related to the unique-maximum chromatic number of the corresponding graph (see [14]).

The other type of vertex coloring can be seen as a relaxation of the unique-maximum coloring.

Definition 1.2. A *conflict-free coloring with respect to paths* of $G = (V, E)$ with k colors is a function $C\colon V \to \{1, \ldots, k\}$ such that for each path p in G there is a color that occurs exactly once on the vertices of p. The minimum k for which a graph G has a conflict-free coloring with k colors is called the *conflict-free chromatic number* of G and is denoted by $\chi_{\mathrm{cf}}(G)$.

Conflict-free coloring of graphs with respect to paths is a special case of conflict-free colorings of hypergraphs, studied in Even et al. [8] and Smorodinsky [18]. One of the applications of conflict-free colorings is that it represents a frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile agents*. Base stations have fixed positions and provide the backbone of the network; they are represented by vertices in V. Mobile agents are the clients of the network and they are served by base stations. This is done as follows: Every base station has a fixed frequency; this is represented by the coloring C, i.e., colors represent frequencies. If an agent wants to establish a link with a base station it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not used by some other base station in the range. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, where possible, is preferable. Conflict-free coloring problems have been the subject of many recent papers due to their practical and theoretical interest (see e.g. [15, 9, 6, 7, 3]). Most approaches in the conflict-free coloring literature use unique-maximum colorings (a notable exception is the 'triples' algorithm in [3]), because unique-maximum colorings are easier to argue about in proofs, due to their additional structure. Another advantage of unique-maximum colorings is the simplicity of computing the unique color in any range (it is always the maximum color), given a unique-maximum coloring, which can be helpful if very simple mobile devices are used by the agents.

For general graphs, finding the exact unique-maximum chromatic number of a graph is NP-complete [17, 14] and there is a polynomial time $O(\log^2 n)$ approximation algorithm [5], where n is the number of vertices. Since the problem is hard in general, it makes sense to study specific graphs.

The $m \times m$ *grid*, G_m, is the *cartesian product* of two paths, each of length $m - 1$, that is, the vertex set of G_m is $\{0, \ldots, m-1\} \times \{0, \ldots, m-1\}$ and the edges are $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| \le 1\}$. It is known [11] that for general planar graphs the unique-maximum chromatic number is $O(\sqrt{n})$. Grid graphs are planar and therefore the $O(\sqrt{n})$ bound applies. One might expect that, since the grid has a relatively simple and regular structure, it should not be hard to calculate its unique-maximum chromatic number. This is why it is rather striking that, even though it is not hard to show upper and lower bounds that are only a small constant multiplicative factor apart, the *exact* value of these chromatic numbers is not known, and has been the subject of [1, 2].

Paper organization. In the rest of this section we provide the necessary definitions and some earlier results. In section 2, we prove that it is coNP-complete to decide whether a given vertex coloring of a graph is conflict-free with respect to paths. In section 3, we show that for every graph $\chi_{\mathrm{um}}(G) \leq 2^{\chi_{\mathrm{cf}}(G)} - 1$ and provide a sequence of graphs for which the ratio $\chi_{\mathrm{um}}(G)/\chi_{\mathrm{cf}}(G)$ tends to 2. In section 4, we introduce two games on graphs that help us relate the two chromatic numbers for the square grid graph. In section 5, we show a lower bound on the unique-maximum chromatic number of the square grid graph, improving previous results. Conclusions and open problems are presented in section 6.

1.1 Preliminaries

Definition 1.3. A graph X is a *minor* of Y, denoted as $X \preccurlyeq Y$, if X can be obtained from Y by a sequence of the following three operations: vertex deletion, edge deletion, and edge contraction. Edge contraction is the process of merging both endpoints of an edge into a new vertex, which is connected to all vertices adjacent to the two endpoints. Given a unique-maximum coloring C of Y, we get the *induced coloring* of X as follows. Take a sequence of vertex deletions, edge deletions, and edge contractions so that we obtain X from Y. For the vertex and edge deletion operations, just keep the colors of the remaining vertices. For the edge contraction operation, say along edge xy, which gives rise to the new vertex v_{xy}, set $C'(v_{xy}) = \max(C(x), C(y))$, and keep the colors of all other vertices.

Proposition 1.4. [4] *If $X \preccurlyeq Y$, and C is a unique-maximum coloring of Y, then the induced coloring C' is a unique-maximum coloring of X. Consequently, $\chi_{\mathrm{um}}(X) \leq \chi_{\mathrm{um}}(Y)$.*

The (traditional) chromatic number of a graph is denoted by $\chi(G)$ and is the smallest number of colors in a vertex coloring for which adjacent vertices are assigned different colors. A simple relation between the chromatic numbers we have defined so far is the following.

Proposition 1.5. *For every graph G, $\chi(G) \leq \chi_{\mathrm{cf}}(G) \leq \chi_{\mathrm{um}}(G)$.*

Proof. Since every unique-maximum coloring is also a conflict-free coloring, we have $\chi_{\mathrm{cf}}(G) \leq \chi_{\mathrm{um}}(G)$. A traditional coloring can be defined as a coloring in which paths of length one are conflict-free. Therefore every conflict-free coloring is also a traditional coloring and thus $\chi(G) \leq \chi_{\mathrm{cf}}(G)$. $\square$

Moreover, we prove that both conflict-free and unique-maximum chromatic numbers are monotone under taking subgraphs.

Proposition 1.6. *If $X \subseteq Y$, then $\chi_{\mathrm{cf}}(X) \leq \chi_{\mathrm{cf}}(Y)$ and $\chi_{\mathrm{um}}(X) \leq \chi_{\mathrm{um}}(Y)$.*

Proof. Take the restriction of any conflict-free or unique-maximum coloring of graph Y to the vertex set $V(X)$. This is a conflict-free or unique maximum coloring of graph X, respectively, because the set of paths of graph X is a subset of all paths of Y. $\square$

If v is a vertex (resp. S is a set of vertices) of graph $G = (V, E)$, denote by $G - v$ (resp. $G - S$) the graph obtained from G by deleting vertex v (resp. vertices of S) and adjacent edges.

Definition 1.7. A subset $S \subseteq V$ is a *separator* of a connected graph $G = (V, E)$ if $G - S$ is disconnected or empty. A separator S is *minimal* if no proper subset $S' \subset S$ is a separator.

2 Deciding whether a coloring is conflict-free

In this section, we show a difference between the two chromatic numbers χ_{um} and χ_{cf}, from the computational complexity aspect. For the notions of complexity classes, hardness, and completeness, we refer, for example, to [16].

As we mentioned before, in [17, 14], it is shown that computing χ_{um} for general graphs is NP-complete. To be exact the following problem is NP-complete: "Given a graph G and an integer k, is it true that $\chi_{\mathrm{um}}(G) \leq k$?". The above fact implies that it is possible to check in polynomial time whether a given coloring of a graph is unique-maximum with respect to paths. We remark that both the conflict-free and the unique-maximum properties have to be true in every path of the graph. However, a graph with n vertices can have exponential in n number of distinct sets of vertices, each one of which is a vertex set of a path in the graph. For unique-maximum colorings we can find a shortcut as follows: Given a (connected) graph G and a vertex coloring of it, consider the set of vertices S of unique colors. Let $u, v \in V \setminus S$ such that they both have the maximum color that appears in $V \setminus S$. If there is a path in $G - S$ from u to v, then this path violates the unique maximum property. Therefore, S has to be a separator in G, which can be checked in polynomial time, otherwise the coloring is not unique-maximum. If $G - S$ is not empty, we can proceed analogously for each of its components. For conflict-free colorings there is no such shortcut, unless coNP = P, as the following theorem implies.

Theorem 2.1. *It is coNP-complete to decide whether a given graph and a vertex coloring of it is conflict-free with respect to paths.*

Proof. In order to prove that the problem is coNP-complete, we prove that it is coNP-hard and also that it belongs to coNP.

We show coNP-hardness by a reduction from the complement of the Hamiltonian path problem. For every graph G, we construct in polynomial time a graph G^* of polynomial size together with a coloring C of its vertices such that G has no Hamiltonian path if and only if C is conflict-free with respect to paths of G^*.

Assume the vertices of graph G are v_1, v_2, ..., v_n. Then, graph G^* consists of two isomorphic copies of G, denoted by $\hat{G}$ and $\check{G}$, with vertex sets $\overline{v}_1$, $\overline{v}_2$, ..., $\overline{v}_n$ and $\underline{v}_1$, $\underline{v}_2$, ..., $\underline{v}_n$, respectively. Additionally, for every $1 \leq i \leq n$, G^* contains the path

$$P_i = \overline{v}_i, v_{i,1}, v_{i,2}, \ldots, v_{i,i-1}, v_{i,i+1}, \ldots, v_{i,n}, \underline{v}_i,$$

where, for every i, $v_{i,1}, v_{i,2}, \ldots, v_{i,i-1}, v_{i,i+1}, \ldots, v_{i,n}$ are new vertices. We use the following notation for the two possible directions to traverse this path:

$$P_i^{\downarrow} = (v_{i,1}, \ldots, v_{i,i-1}, v_{i,i+1}, \ldots, v_{i,n}),$$

$$P_i^{\uparrow} = (v_{i,n}, \ldots, v_{i,i+1}, v_{i,i-1}, \ldots, v_{i,1}).$$

We call paths P_i *connecting paths.*

We now describe the coloring of $V(G^*)$. For every i, we set $C(\overline{v}_i) = C(\underline{v}_i) = i$. For every $i > j$, we set $C(v_{i,j}) = C(v_{j,i}) = n + \binom{i-1}{2} + j$. Observe that every color occurs exactly in two vertices of G^*.

If G has a Hamiltonian path, say $v_1 v_2 \ldots v_n$, then there is a path through all vertices of G^*, either

$$\overline{v}_1 P_1^{\downarrow} \underline{v}_1 \underline{v}_2 P_2^{\uparrow} \overline{v}_2 \ldots \overline{v}_{n-1} P_{n-1}^{\downarrow} \underline{v}_{n-1} \underline{v}_n P_n^{\uparrow} \overline{v}_n, \text{ if } n \text{ is even,}$$

or

$$\overline{v}_1 P_1^{\downarrow} \underline{v}_1 \underline{v}_2 P_2^{\uparrow} \overline{v}_2 \ldots \underline{v}_{n-1} P_{n-1}^{\uparrow} \overline{v}_{n-1} \overline{v}_n P_n^{\downarrow} \underline{v}_n, \text{ if } n \text{ is odd.}$$

But then, this path has no uniquely occurring color and thus C is not conflict-free.

Suppose now that C is not a conflict-free coloring. We prove that G has a Hamiltonian path.

By the assumption, there is a path P in G^* which is not conflict-free. This path must contain none or both vertices of each color. Therefore, P can not be completely contained in $\hat{G}$, or in $\check{G}$, or in some P_i. Also, P can not contain only one of $\overline{v}_i$ and $\underline{v}_i$, for some i. Therefore, P must contain both $\overline{v}_i$ and $\underline{v}_i$ for a non-empty subset of indices i.

Then, it must contain completely some P_i, because vertices in $\hat{G}$ and $\check{G}$ can only be connected with some complete P_i. But since each one of the $n-1$ colors of this P_i occurs in a different connecting paths, P must contain a vertex in every connecting path. But then P must contain every $\overline{v}_i$ and $\underline{v}_i$, because vertices in P_i can only be connected to the rest of the graph through one of $\overline{v}_i$ or $\underline{v}_i$.

Suppose that P is not a Hamiltonian path of G^*. Observe that if P does not contain all vertices of some connecting path P_i, then one of its end vertices should be there. If P does not contain vertex $v_{i,j}$, then it can not contain $v_{j,i}$ either. But then one end vertex of P should be on P_i, the other one on P_j, and all other vertices of G^* are on P. Therefore, we can extend P such that it contains $v_{i,j}$ and $v_{j,i}$ as well. So assume in the sequel that P is a Hamiltonian path of G^*.

Now we modify P, if necessary, so that both of its end-vertices e and f lie in $V(\hat{G}) \cup V(\check{G})$. If e and f are adjacent in G^*, then add the edge ef to P and we get a Hamiltonian cycle of G^*. Now remove one of its edges which is either in $\hat{G}$, or in $\check{G}$ and get the desired Hamiltonian path. Suppose now that e and f are not adjacent, and e is on one of the connecting paths. Then e should be adjacent to the end vertex e' of that connecting path, which is in $\hat{G}$ or in $\check{G}$. Add edge ee' to P. We get a cycle and a path joined in e'. Remove the other edge of the cycle adjacent to e'. We have a Hamiltonian path now, whose end vertex is e' instead of e. Proceed analogously for f, if necessary.

Now we have a Hamiltonian path P of G^* with end-vertices in $V(\hat{G}) \cup V(\check{G})$. Then, P is in the form, say,

$$\overline{v}_1 P_1^{\downarrow} \underline{v}_1 \underline{v}_2 P_2^{\uparrow} \overline{v}_2 \ldots \overline{v}_{n-1} P_{n-1}^{\downarrow} \underline{v}_{n-1} \underline{v}_n P_n^{\uparrow} \overline{v}_n, \text{ if } n \text{ is even,}$$

or

$$\overline{v}_1 P_1^{\downarrow} \underline{v}_1 \underline{v}_2 P_2^{\uparrow} \overline{v}_2 \ldots \underline{v}_{n-1} P_{n-1}^{\uparrow} \overline{v}_{n-1} \overline{v}_n P_n^{\downarrow} \underline{v}_n, \text{ if } n \text{ is odd.}$$

But then, $v_1 v_2 \ldots v_n$ is a Hamiltonian path in G.

Finally, the problem is in coNP because one can verify that a coloring of a given graph is not conflict-free in polynomial time, by giving the corresponding path. □

We show an example graph G, its transformation graph G^*, and its coloring C in figure 1.

3 The two chromatic numbers of general graphs

We have seen that $\chi_{\mathrm{um}}(G) \geq \chi_{\mathrm{cf}}(G)$ (proposition 1.5). In this section we show that $\chi_{\mathrm{um}}(G)$ can not be larger than an exponential function of $\chi_{\mathrm{cf}}(G)$. We also provide an infinite sequence of graphs $H_1, H_2, \ldots,$ for which $\lim_{k\to\infty}(\chi_{\mathrm{um}}(H_k)/\chi_{\mathrm{cf}}(H_k)) = 2$.

The path of n vertices is denoted by P_n. It is known that $\chi_{\mathrm{um}}(P_n) = \lfloor \log_2 n \rfloor + 1$ (see for example [8]).

Lemma 3.1. *For every path P_n, $\chi_{\mathrm{cf}}(P_n) = \lfloor \log_2 n \rfloor + 1$.*

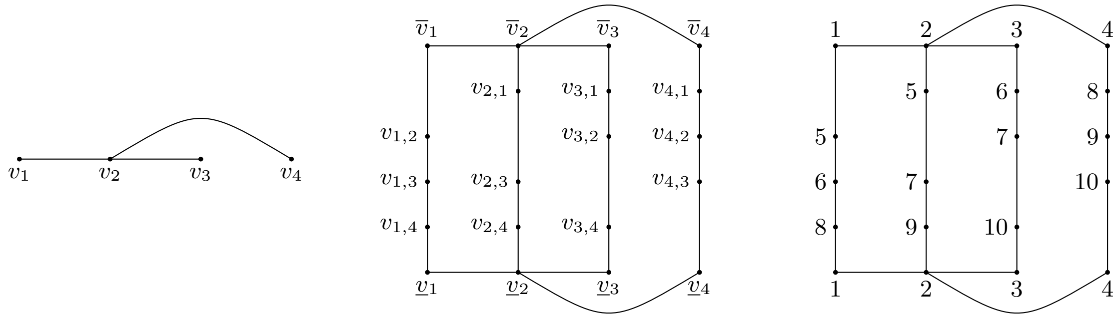

Figure 1: Example graphs G, G^*, and coloring C of G^*

Proof. By proposition 1.5, $\chi_{\rm cf}(P_n) \le \chi_{\rm um}(P_n)$. We prove a matching lower bound by induction. We have $\chi_{\rm cf}(P_1) \ge 1$. For $n > 1$, there is a uniquely occurring color in any conflict-free coloring of the the whole path P_n. Then, $\chi_{\rm cf}(P_n) \ge 1 + \chi_{\rm cf}(P_{\lfloor n/2 \rfloor})$, which implies $\chi_{\rm cf}(P_n) \ge \lfloor \log_2 n \rfloor + 1$. $\square$

Moreover, we are going to use the following result (lemma 5.1 of [11]): If the longest path of G has k vertices, then $\chi_{\rm um}(G) \le k$.

Proposition 3.2. *For every graph G, $\chi_{\rm um}(G) \le 2^{\chi_{\rm cf}(G)} - 1$.*

Proof. Set $j = \chi_{\rm cf}(G)$. For any path $P \subseteq G$, $\chi_{\rm cf}(P) \le j$, therefore, by lemma 3.1, the longest path has at most $2^j - 1$ vertices, so by lemma 5.1 of [11], $\chi_{\rm um}(G) \le 2^j - 1$. $\square$

We define recursively the following sequence of graphs: Graph H_0 is a single vertex. Suppose that we have already defined H_{k-1}. Then H_k consists of (a) a $K_{2^{k+1}-1}$, (b) $2^{k+1} - 1$ copies of H_{k-1}, and (c) for for each i, $1 \le i \le 2^{k+1} - 1$, the i-th vertex of the $K_{2^{k+1}-1}$ is connected by an edge to one of the vertices of the i-th copy of H_{k-1}.

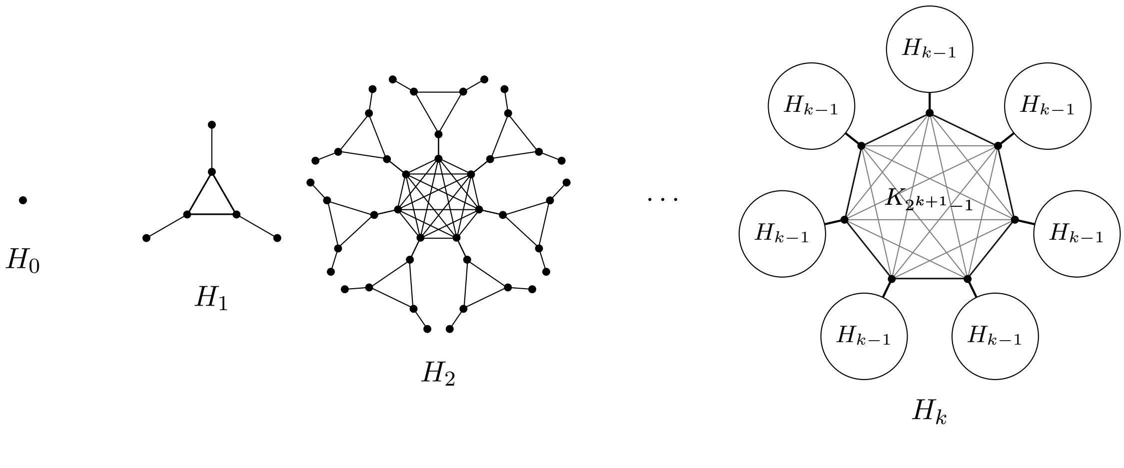

Figure 2: Sequence of graphs

Lemma 3.3. *For $k \ge 0$, $\chi_{\rm cf}(H_k) = 2^{k+1} - 1$.*

Proof. By induction on k. For $k = 0$, $\chi_{\rm cf}(H_0) = 1$. For $k > 0$, we have $H_k \supseteq K_{2^{k+1}-1}$, therefore, $\chi_{\rm cf}(H_k) \ge 2^{k+1} - 1$.

In order to prove that $\chi_{\mathrm{cf}}(H_k) \le 2^{k+1}-1$, it is enough to describe a conflict-free coloring of H_k with $2^{k+1}-1$ colors, given a conflict-free coloring of H_{k-1} with 2^k-1 colors: We color the vertices of the clique $K_{2^{k+1}-1}$ with colors $1, 2, \ldots, 2^{k+1}-1$ such that the i-th vertex is colored with color i. Consider these colors mod $2^{k+1}-1$, e. g. color 2^{k+1} is identical to color 1. Recall that the i-th copy of H_{k-1} has a vertex connected to the i-th vertex of $K_{2^{k+1}-1}$, and by induction we know that $\chi_{\mathrm{cf}}(H_{k-1}) = 2^k-1$. Color the i-th copy of H_{k-1}, with colors $i+1, i+2, \ldots, i+2^k-1$.

We claim that this vertex coloring of H_k is conflict-free. If a path is completely contained in a copy of H_{k-1}, then it is conflict-free by induction. If a path is completely contained in the clique $K_{2^{k+1}-1}$, then it is also conflict-free, because all colors in the clique part are different. If a path contains vertices from a single copy of H_{k-1}, say, the i-th copy, and the clique, then the i-th vertex of the clique is on the path and uniquely colored. The last case is when a path contains vertices from exactly two copies of H_{k-1}. Suppose that these are the i-th and j-th copies of H_{k-1}, $1 \le i < j \le 2^{k+1}-1$. If $i+2^k-1 < j$, then color j is unique in the path; indeed, the i-th copy of H_{k-1} is colored with colors $i+1, \ldots i+2^k-1$, and the j-th copy of H_{k-1} is colored with colors $j+1, \ldots j+2^k-1$, while color j appears only once in $K_{2^{k+1}-1}$. Similarly, if $i+2^k-1 \ge j$, then color i is unique in the path. □

Lemma 3.4. *$\chi_{\mathrm{um}}(H_k) \le 2^{k+2}-k-3$.*

Proof. By induction. For $k=0$, $\chi_{\mathrm{um}}(H_0)=1$. For $k>0$, in order to color H_k use the $2^{k+1}-1$ different highest colors for the clique part. By the inductive hypothesis $\chi_{\mathrm{um}}(H_{k-1}) \le 2^{k+1}-k-2$. For each copy of H_{k-1}, use the same coloring with the $2^{k+1}-k-2$ lowest colors. This coloring of H_k is unique maximum. Indeed, if a path is contained in a copy of H_{k-1} then it is unique maximum by induction, and if it contains a vertex in the clique part, then it is also unique maximum. The total number of colors is $2^{k+2}-k-3$. □

Lemma 3.5. *If Y is a graph that consists of a K_ℓ and ℓ isomorphic copies of a connected graph X, such that for $1 \le i \le \ell$ a vertex of it i-th copy is connected to the i-th vertex of K_ℓ by an edge. Then we have $\chi_{\mathrm{um}}(Y) \ge \ell - 1 + \chi_{\mathrm{um}}(X)$*

Proof. By induction on ℓ. For $\ell = 1$, we have that $\chi_{\mathrm{um}}(Y) \ge \chi_{\mathrm{um}}(X)$, because $Y \supseteq X$. For the inductive step, for $\ell > 1$, if Y consists of a K_l and ℓ copies of X, then Y is connected, and thus contains a vertex v with unique color. But then, $Y - v \supseteq Y'$, where Y' is a graph that consists of a $K_{\ell-1}$ and $\ell-1$ isomorphic copies of a X, each connected to a different vertex of $K_{\ell-1}$, and thus $\chi_{\mathrm{um}}(Y) = 1 + \chi_{\mathrm{um}}(Y') \ge \ell - 1 + \chi_{\mathrm{um}}(X)$. □

Lemma 3.6. *$\chi_{\mathrm{um}}(H_k) \ge 2^{k+2}-2k-3$.*

Proof. By induction. For $k=0$, $\chi_{\mathrm{um}}(H_0)=1$. For $k>0$, by the inductive hypothesis and lemma 3.5, $\chi_{\mathrm{um}}(H_k) \ge 2^{k+1}-1-1+2^{k+1}-2(k-1)-3 = 2^{k+2}-2k-3$ □

Theorem 3.7. *We have $\lim_{k\to\infty}(\chi_{\mathrm{um}}(H_k)/\chi_{\mathrm{cf}}(H_k)) = 2$.*

Proof. From lemmas 3.3, 3.4, 3.6, we have

$$\frac{2^{k+2}-2k-3}{2^{k+1}-1} \le \frac{\chi_{\mathrm{um}}(H_k)}{\chi_{\mathrm{cf}}(H_k)} \le \frac{2^{k+2}-k-3}{2^{k+1}-1}$$

which implies that the ratio tends to 2. □

4 The two chromatic numbers of a square grid

In this section, we define two games on graphs, each played by two players. The first game characterizes completely the unique-maximum chromatic number of the graph. The second game is related to the conflict-free chromatic number of the graph. We use the two games to prove that the conflict-free chromatic number of the square grid is a function of the unique-maximum chromatic number of the square grid. This is useful because it allows to translate existing lower bounds on the unique-maximum chromatic number of the square grid to lower bounds on the corresponding conflict-free chromatic number. For any graph G, and subset of its vertices $V' \subset V(G)$, let $G[V']$ denote the subgraph of G induced by V'.

The first game (which is played on a graph G by two players) is the *connected component game*:

$i \leftarrow 0$; $G^0 \leftarrow G$
while $V(G^i) \neq \emptyset$:
  increment i by 1
  Player 1 chooses a connected component S^i of G^{i-1}
  Player 2 chooses a vertex $v_i \in S^i$
  $G^i \leftarrow G^{i-1}[S^i \setminus \{v_i\}]$

The game is finite, because if G^i is not empty, then G^{i+1} is a strict subgraph of G^i. The result of the game is its length, that is, the final value of i. Player 1 tries to make the final value of i as large as possible and thus is the maximizer player. Player 2 tries to make the final value of i as small as possible and thus is the minimizer player. If both players play optimally, then the result is the *value* of the connected component game on graph G, which is denoted by $v^{\text{cs}}(G)$.

Proposition 4.1. *In the connected component game, there is a strategy for player 2 (the minimizer), so that the result of the game is at most $\chi_{\text{um}}(G)$, i.e., $v^{\text{cs}}(G) \leq \chi_{\text{um}}(G)$.*

Proof. By induction on $\chi_{\text{um}}(G)$: If $\chi_{\text{um}}(G) = 0$, i.e., the graph is empty, the value of the game is 0. If $\chi_{\text{um}}(G) = k > 0$, then in the first turn some connected component S_1 is chosen by player 1. Then, the strategy of player 2 is to take an optimal unique-maximum coloring C of G and choose a vertex v_1 in S^1 that has a unique color in S^1. Then, $G^1 = G[S^1 \setminus \{v_1\}] \subset G^0$ and the restriction of C to $S^1 \setminus \{v_1\}$ is a unique-maximum coloring of G^1 that is using at most $k-1$ colors. Thus, $\chi_{\text{um}}(G^1) \leq k-1$, and by the inductive hypothesis player 2 has a strategy so that the result of the game on G^1 is at most $k-1$. Therefore, player 2 has a strategy so that the result of the game on $G^0 = G$ is at most $1 + k - 1 = k$. ☐

Lemma 4.2. *For every $v \in V(G)$, $\chi_{\text{um}}(G - v) \geq \chi_{\text{um}}(G) - 1$*

Proof. Assume for the sake of contradiction that there exists a $v \in V(G)$ for which $\chi_{\text{um}}(G - v) < \chi_{\text{um}}(G) - 1$. Then an optimal coloring of $G - v$ can be extended to a coloring of G, where v has a new unique maximum color. Therefore there is a coloring of G that uses less than $\chi_{\text{um}}(G) - 1 + 1 = \chi_{\text{um}}(G)$ colors; a contradiction. ☐

Proposition 4.3. *In the connected component game, there is a strategy for player 1 (the maximizer), so that the result of the game is at least $\chi_{\text{um}}(G)$, i.e., $v^{\text{cs}}(G) \geq \chi_{\text{um}}(G)$.*

Proof. By induction on $\chi_{\text{um}}(G)$: If $\chi_{\text{um}}(G) = 0$, i.e., the graph is empty, the result of the game is zero. If $\chi_{\text{um}}(G) = k > 0$, the strategy of player 1 is to choose a connected component S^1 such that $\chi_{\text{um}}(G[S^1]) = k$. For every choice of v_1 by Player 2, by lemma 4.2, $\chi_{\text{um}}(G^1) \geq k-1$, and thus, by the inductive hypothesis player 1 has a strategy so that the result of the game on G^1 is at least $k-1$. Therefore, the result of the game on $G^0 = G$ is at least $1 + k - 1 = k$. ☐

Corollary 4.4. *For every graph, $v^{\mathrm{cs}}(G) = \chi_{\mathrm{um}}(G)$.*

The second game (also played on a graph G by two players) is the *path game*:

$i \leftarrow 0$; $G^0 \leftarrow G$
while $V(G^i) \neq \emptyset$:
  increment i by 1
  Player 1 chooses the set of vertices S^i of a path of G^{i-1}
  Player 2 chooses a vertex $v_i \in S^i$
  $G^i \leftarrow G^{i-1}[S^i \setminus \{v_i\}]$

The only difference with the connected component game is that in the path game the vertex set S^i that maximizer chooses is the vertex set of a path of the graph G^{i-1}. If both players play optimally, then the result is the *value* of the path game on graph G, which is denoted by $v^{\mathrm{p}}(G)$.

Proposition 4.5. *In the path game, there is a strategy for player 2 (the minimizer), so that the result of the game is at most $\chi_{\mathrm{cf}}(G)$, i.e., $v^{\mathrm{p}}(G) \leq \chi_{\mathrm{cf}}(G)$.*

Proof. By induction on $\chi_{\mathrm{cf}}(G)$: If $\chi_{\mathrm{cf}}(G) = 0$, i.e., the graph is empty, the value of the game is 0. If $\chi_{\mathrm{cf}}(G) = k > 0$, then in the first turn some vertex set S^1 of a path of G is chosen by player 1. Then, the strategy of player 2 is to find an optimal conflict-free coloring C of G and choose a vertex v_1 in S^1 that has a unique color in S^1. Then, $G^1 = G[S^1 \setminus \{v_1\}] \subset G^0$ and the restriction of C to $S^1 \setminus \{v_1\}$ is a conflict-free coloring of G^1 that is using at most $k-1$ colors. Thus, $\chi_{\mathrm{cf}}(G^1) \leq k-1$, and by the inductive hypothesis player 2 has a strategy so that the result of the game is at most $k-1$. Therefore, player 2 has a strategy so that the result of the game is at most $1+k-1 = k$. $\square$

A proposition analogous to 4.3 for the path game is not true. For example, for the complete binary tree of four levels (with 15 vertices, 8 of which are leaves), B_4, it is not difficult to check that $v^{\mathrm{p}}(B_4) = v^{\mathrm{p}}(P_7) = 3$, but $\chi_{\mathrm{cf}}(B_4) = 4$.

Now, we are going to concentrate on the square grid graph. Assume that m is even. We intend to translate a strategy of player 1 (the maximizer) on the connected component game for graph $G_{m/2}$ to a strategy for player on the path game for graph G_m.

Observe that for every connected graph G, there is an ordering of its vertices, $v_1, v_2, \ldots, v_n$ such that the subgraph induced by the first k vertices (for every $1 \leq k \leq n$) is also connected. Just pick a vertex to be v_1, and add the other vertices one by one such that the new vertex v_i is connected to the graph induced by $v_1, \ldots, v_{i-1}$. This is possible, since G itself is connected. We call such an ordering of the vertices an *always-connected ordering.*

Now we decompose the vertex set of G_m into groups of four vertices,

$$Q_{x,y} = \{(2x, 2y), (2x+1, 2y), (2x, 2y+1), (2x+1, 2y+1)\},$$

for $0 \leq x, y < m/2$, called *special quadruples*, or briefly quadruples. We denote the set of quadruples with $W_m = \{Q_{x,y} \mid 0 \leq x, y < m/2\}$ and let $\tau(x, y) = Q_{x,y}$ be a bijection between vertices of $V(G_{m/2})$ and W_m. Extend τ for subsets of vertices of $G_{m/2}$ in a natural way, for any $S \subseteq V(G_{m/2})$, $\tau(S) = \bigcup_{(x,y)\in S} \tau(x, y)$. Define also a kind of inverse τ' of τ as $\tau'(x, y) = (\lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ for any $0 \leq x, y < m$, and for any $S \subseteq V(G_m)$, $\tau'(S) = \{\tau'(x, y) \mid (x, y) \in S\}$.

Let $(x, y) \in V(G_{m/2})$. We call vertices $(x, y+1)$, $(x, y-1)$, $(x-1, y)$, and $(x+1, y)$, if they exist, the *upper*, *lower*, *left*, and *right neighbors* of (x, y), respectively. Similarly, quadruples $Q_{x,y+1}$, $Q_{x,y-1}$, $Q_{x-1,y}$, and $Q_{x+1,y}$ the *upper*, *lower*, *left*, and *right neighbors* of $Q_{x,y}$, respectively.

Quadruple $Q_{x,y}$ induces four edges in G_m, $\{(2x+1, 2y), (2x+1, 2y+1)\}$, $\{(2x, 2y), (2x, 2y+1)\}$, $\{(2x, 2y), (2x, 2y+1)\}$, $\{(2x+1, 2y), (2x+1, 2y+1)\}$, we call them *upper*, *lower*, *left*, and *right* edges of $Q_{x,y}$.

By *direction* d, we mean one of the four basic directions, *up*, *down*, *left*, *right*. For a given set $S \subseteq V(G_{m/2})$, we say that $v \in S$ is *open* in S in direction d, if its neighbor in direction d is not in S. In this case we also say that $\tau(v)$ is open in $\tau(S)$ in direction d.

Lemma 4.6. *If S induces a connected subgraph in $G_{m/2}$, then there is a path in G_m whose vertex set is $\tau(S)$.*

Proof. We prove a stronger statement: If S induces a connected subgraph in $G_{m/2}$, then there is a cycle C in G_m whose vertex set is $\tau(S)$, and if $v \in S$ is open in direction d in S, then C contains the d-edge of $\tau(v)$.

The proof is by induction on $|S| = k$. For $k = 1$, $\tau(S)$ is one quadruple and we can take its four edges.

Suppose that the statement has been proved for $|S| < k$, and assume that $|S| = k$. Consider an always-connected ordering $v_1, v_2, \ldots, v_k$ of S. Let $S' = S \setminus v_k$. By the induction hypothesis, there is a cycle C' satisfying the requirements. Vertex v_k has at least one neighbor in S', say, v_k is the neighbor of v_i in direction d. But then, v_i is open in direction d in S', therefore, C' contains the d-edge of $\tau(v_i)$. Remove this edge from C' and substitute by a path of length 5, passing through all four vertices of $\tau(v_k)$. The resulting cycle, C, contains all vertices of $\tau(S)$, it contains each edge of $\tau(v_k)$, except the one in the opposite direction to d, and it contains all edges of C', except the d-edge of $\tau(v_k)$, but v_k is not open in S in direction d. This concludes the induction step, and the proof. $\square$

Proposition 4.7. *For every $m > 1$, $v^{\mathrm{p}}(G_m) \geq v^{\mathrm{cs}}(G_{\lfloor m/2 \rfloor})$.*

Proof. Assume, without loss of generality that m is even (if not work with graph G_{m-1} instead). In order, to prove that $v^{\mathrm{p}}(G_m) \geq v^{\mathrm{cs}}(G_{\lfloor m/2 \rfloor})$ it is enough, given a strategy for player 1 in the connected set game for $G_{m/2}$, to construct a strategy for player 1 (the maximizer) in the path game for G_m, so that the result of the path game is at least as much as the result of the connected set game. We present the argument as if player 1, apart from the path game, plays in parallel a connected set game on $G_{m/2}$ (for which player 1 has a given strategy to choose connected sets in every round), where player 1 also chooses the moves of player 2 in the connected set game.

At round i of the path game on G_m, player 1 simulates round i of the connected set game on $G_{m/2}$. At the start of round i, player 1 has a graph $G^{i-1} \subseteq G_m$ in the path game and a graph $\hat{G}^{i-1} \subseteq G_{m/2}$ in the connected set game. Player 1 chooses a set $\hat{S}^i$ in the simulated connected set game from his given strategy, and then constructs the path-spanned set $S^i = \tau(\hat{S}^i)$ (by lemma 4.6) and plays it in the path game. Then player 2 chooses a vertex $v_i \in S^i$. Player 1 computes $\hat{v}_i = \tau'(v_i)$ and simulates the move $\hat{v}_i$ of player 2 in the connected set game. This is a legal move for player 2 in the connected set game because $\hat{v}_i \in \hat{S}^i$.

We just have to prove that $S^i = \tau(\hat{S}^i)$ is a legal move for player 1 in the path game, i.e., $S^i \subseteq V(G^{i-1})$. We also have to prove $S^i = \tau(\hat{S}^i)$ is spanned by a path in G^{i-1} but this is always true by lemma 4.6, since $\hat{S}^i$ is a connected vertex set in $\hat{G}^{i-1}$. Since $S^i \subseteq \tau(V(\hat{G}^{i-1}))$, it is enough to prove that at round i, $\tau(V(\hat{G}^{i-1})) \subseteq V(G^{i-1})$. The proof is by induction on i. For $i = 1$, $G^0 = G_m$, $\hat{G}^0 = G_{m/2}$, and thus $\tau(V(\hat{G}^0)) = V(G^0)$. At the start of round i with $i > 1$, $\tau(V(\hat{G}^{i-1})) \subseteq V(G^{i-1})$, by the inductive hypothesis. Then, $\tau(\hat{S}^i) = S^i$ and $\tau(\hat{S}^i \setminus \{\hat{v}_i\}) = \tau(\hat{S}^i) \setminus \tau(\hat{v}_i) = S^i \setminus \tau(\hat{v}_i) \subseteq S^i \setminus \{v_i\}$, because $v_i \in \tau(\hat{v}_i)$. Thus, $\tau(V(\hat{G}^{i-1}[\hat{S}^i \setminus \{\hat{v}_i\}])) \subseteq V(G^{i-1}[S^i \setminus \{v_i\}])$, i.e., $\tau(V(\hat{G}^i)) \subseteq V(G^i)$. $\square$

Theorem 4.8. *For every $m > 1$, $\chi_{\mathrm{cf}}(G_m) \geq \chi_{\mathrm{um}}(G_{\lfloor m/2 \rfloor})$.*

Proof. By proposition 4.5, $\chi_{\mathrm{cf}}(G_m) \geq v^{\mathrm{p}}(G_m)$, by proposition 4.7, $v^{\mathrm{p}}(G_m) \geq v^{\mathrm{cs}}(G_{\lfloor m/2 \rfloor})$, and by proposition 4.3, $v^{\mathrm{cs}}(G_{\lfloor m/2 \rfloor}) \geq \chi_{\mathrm{um}}(G_{\lfloor m/2 \rfloor})$. □

5 Lower bounds on the chromatic numbers of the square grid

Recall that G_m is the $m \times m$ grid graph, that is, the cartesian product of two paths, each of length $m-1$. It was shown in [2] that $\chi_{\mathrm{um}}(G_m) \geq 3m/2$. The best known upper bound is $\chi_{\mathrm{um}}(G_m) \leq 2.519m$, from [1, 2]. The main result of this section is the following improvement of the lower bound.

Theorem 5.1. *For $m \geq 2$, $\chi_{\mathrm{um}}(G_m) \geq \frac{5}{3}m - 18\log_2 m$.*

Proof. For any subset $A \subset V(G)$, let $N_G(A)$ denote the *boundary* of A, that is, all vertices which are not in A, but neighbors of some vertex in A. Observe that in a unique-maximum coloring of a connected graph G, the set of vertices of unique colors form a separator (see, e.g., [11]). Indeed, remove all vertices of unique colors from G, let G' be the remaining graph and let color c be the highest remaining color. It is not unique, let u and v be two vertices of color c. Then there can not be a path in G' from u to v, therefore, G' is not connected.

We will use induction on m. Consider a unique-maximum coloring of G_m and take a minimal separator, formed by vertices of unique colors. Using the separator and the coloring, after applying a carefully selected sequence of minor operations (vertex deletion, edge deletion, edge contraction) on G_m, we obtain an induced unique-maximum coloring (see definition 1.3) of $G_{m'}$ for some $m' < m$, and we apply the induction hypothesis to prove the lower bound.

Throughout the proof, we consider $G = G_m$ in its *standard drawing*, that is, the vertices are points (x, y), $0 \leq x, y \leq m-1$, two vertices (x, y) and (x', y') are connected if and only if $|x - x'| + |y - y'| = 1$, and edges are drawn as straight line segments. If it is clear from the context, we do not make any notational distinction between vertices (edges) and points (resp. segments) representing them. Denote by V the vertices of the grid, that is, $V = V(G)$. Take an additional vertex v, "outside" G_m, say, at $(-2, -2)$, and connect it with all boundary vertices of G_m, so that we do not create any edge crossing. Let $G' = G'_m$ denote the resulting graph, Let $V' = V(G')$.

Define graph H' and its drawing as follows. The vertex set of H' is V'. Vertex v is connected to the boundary vertices of the grid, just like in G'. Two vertices, (x, y) and (x', y') in the grid are connected by a straight line segment in H' if and only if $|x - x'| \leq 1$ and $|y - y'| \leq 1$.

Suppose that $S \subset V'$, and $H'[S]$ contains a non-self-intersecting cycle C. Let A (resp. B) be those vertices in V' which are *inside* (resp. *outside*) C. If $A, B \neq \emptyset$, then C is called a *separating* cycle. If $A = \emptyset$, then C is called an *empty* cycle. Suppose that C is a separating cycle. Since edges of H' and edges of G' do not intersect each other, S separates A and B in G'.

Suppose now that S is a separator in G' and let A be the vertex set of one of the connected components, separated by S. Clearly, the boundary of A, $N_{G'}(A)$ belongs to S, and an easy case analysis shows that the edges of $H'[N_{G'}(A)]$, in the present drawing, separate the vertices of A from the other vertices. Suppose from now that S is a minimal separator. Then, by the previous observations, $H'[S]$ contains one or more separating cycles. Let C be a separating cycle in $H'[S]$ with the smallest number of points inside, and let A be the set of these points. Then $N_{G'}(A) \subset C$, but since $N_{G'}(A)$ already separates A from the other vertices, $N_{G'}(A) = S$. Observe that the only empty cycle in H' is the right angled triangle with leg 1. If $H'[S]$ contains such a cycle, then one of its vertices can be removed from S and we still have a separator. Therefore, there are no empty cycles in $H'[S]$. Moreover, by the minimality of S, every separating cycle in $H'[S]$ contains exactly the points of A in its interior. It follows, that $H'[S]$ is a cycle that has A in its interior, and the remaining points, $V' \setminus (S \cup A)$ in the exterior.

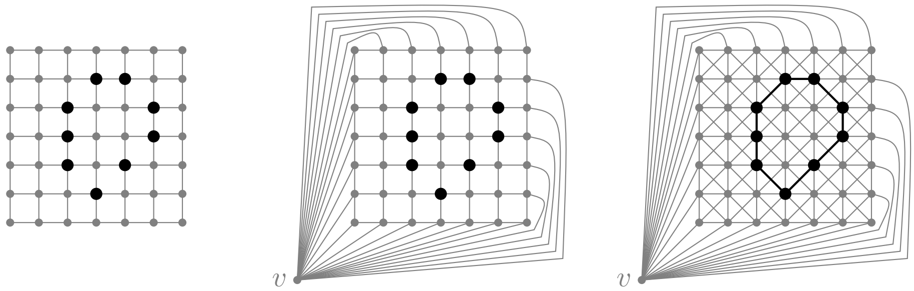

Figure 3: A cycle-separator in G, G', and H', for $m = 7$

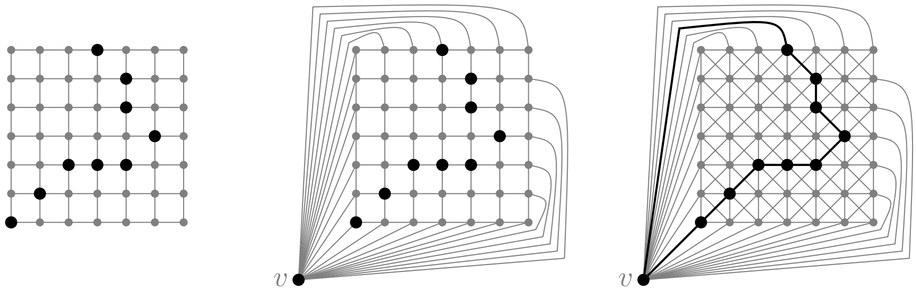

Figure 4: A path-separator in G, G', and H', for $m = 7$

It is easy to see that if S is a separator in G_m, then $S \cup \{v\}$ is a separator in G'_m. On the other hand, if S is a separator is G'_m, then $S \setminus \{v\}$ is a separator in G'_m. Consequently, if S is a minimal separator in G_m, then either S is a minimal separator in G'_m, or $S \cup \{v\}$ is a minimal separator in G'_m. In the first case we say that S is a *cycle-separator* (see figure 3), in the second case we say that it is a *path-separator* (see figure 4) of G_m. The vertices of a cycle-separator form a cycle in H', and the vertices of a path-separator form a path, whose first and last vertices are the only neighbors of v, that is, they are on the boundary of the grid, and the other vertices of S are not on the boundary.

Our bound is negative for $m \le 64$, so assume that $m > 64$, and the statement has been proved for smaller values of m. Consider an optimal coloring of G_m, and let S be a minimal separator, all of whose vertices have unique colors.

Case 1: *S is a cycle-separator.* Let z be the smallest value of $x + y$ over all vertices of S, and let (x, y) be the vertex of S for which $x+y = z$, and y is the largest. Then vertex $(x+1, y-1)$ is also a vertex of S, and one of $(x, y+1)$, $(x+1, y+1)$ is also in S. Let (x', y') be the vertex of S for which $x + y = z$, and y is the smallest. Then $y' < y$, since $(x + 1, y - 1)$ is in S. Moreover, vertex $(x' - 1, y' + 1)$ is also a vertex of S, and one of $(x' + 1, y')$, $(x' + 1, y' + 1)$ is also in S. Consider the following contractions of horizontal edges: $(x, m-1)(x+1, m-1)$, $(x, m-2)(x+1, m-2)$, $\ldots$, $(x, y)(x+1, y)$, $(x+1, y-1)(x+2, y-1)$, $(x+2, y-2)(x+3, y-2)$, $\ldots$, $(x', y')(x'+1, y')$, $(x', y'-1)(x'+1, y'-1)$, $\ldots$, $(x', 0)(x'+1, 0)$, and vertical edges: $(0, y)(0, y+1)$, $(1, y)(1, y+1)$, $\ldots$, $(x, y)(x, y+1)$, $(x+1, y)(x+1, y+1)$, $(x+2, y-1)(x+2, y)$, $\ldots$, $(x'+1, y')(x'+1, y'+1)$, $(x'+2, y')(x'+2, y'+1)$, $\ldots$, $(m-1, y')(m-1, y'+1)$. We obtain a graph, which contains G_{m-1} as a subgraph and the induced coloring uses at least two less colors that the coloring of G_m See figure 5, where for each gray area, vertices are contracted to a single vertex. The induced coloring uses at least $\chi_{\mathrm{um}}(G_{m-1})$ colors, therefore, we have $\chi_{\mathrm{um}}(G_m) \ge \chi_{\mathrm{um}}(G_{m-1}) + 2 \ge \frac{5}{3}(m-1) - 18\log_2(m-1) + 2 > \frac{5}{3}m - 18\log_2 m$.

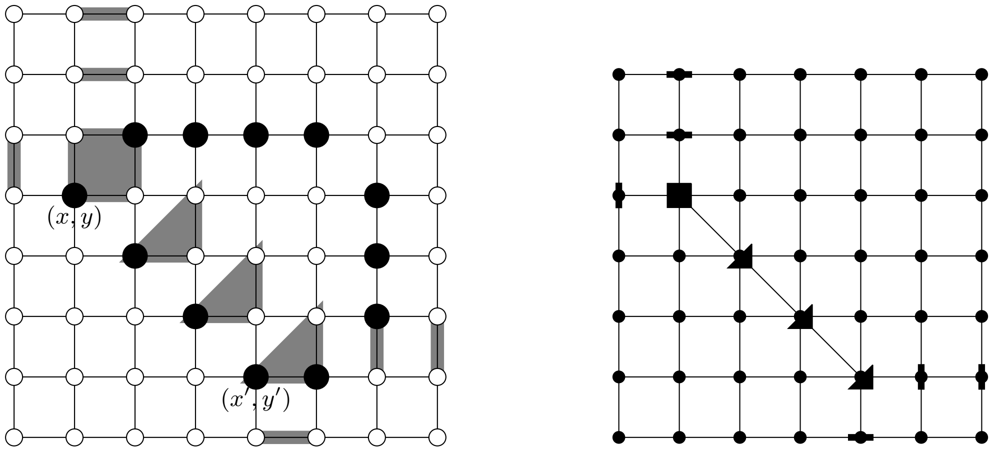

Figure 5: Graph G_m with edge contractions and its minor containing G_{m-1}

Case 2: *S is a path-separator.* By symmetry we can assume that the path starts in column $x = 0$. If it ends in $x = 0$, $y = 0$, or in $y = m-1$, then, we can remove column $x = 0$, and either row $y = 0$ or $y = m-1$, and get a unique maximum coloring of G_{m-1} with at least two less colors. Then we apply induction as in case 1. So we can assume that S ends in $x = m-1$. It follows that $|S| \geq m$. We distinguish two subcases.

Subcase 2.1. *S starts in $x = 0$, ends in $x = m-1$, and $|S| > m$.*
Orient the path formed by the vertices of S. For simplicity, call the oriented path $v_1, \ldots v_{|S|}$ also S. The edges of S can be of eight types, left, right, upper, lower, left-upper, left-lower, right-upper, right-lower.

Suppose first that S contains two edges, one of them is vertical (left or right edge), one of them is horizontal (upper or lower edge), say, $(x, y)(x+1, y)$ and $(x', y')(x', y'+1)$. Then contract all edges $(x, i)(x+1, i)$, and all edges $(i, y')(i, y'+1)$, $0 \leq i \leq m-1$, to obtain G_{m-1}, whose induced coloring uses at most $\chi_{\mathrm{um}}(G_m) - 2$ colors. Therefore, we have $\chi_{\mathrm{um}}(G_m) \geq \chi_{\mathrm{um}}(G_{m-1}) + 2 \geq \frac{5}{3}(m-1) - 18 \log_2(m-1) + 2 > 5m/3 - 18 \log_2 m$. So, we can assume in the sequel that either there are no vertical edges, or no horizontal edges in S. Suppose that there are no *horizontal* edges, and let $v_i = (x, y)$ be a vertex of S where y is the largest. Then $v_{i-1}v_i$ is an upper-right edge, and $v_i v_{i+1}$ is a lower-right edge, or $v_{i-1}v_i$ is an upper-left edge, and $v_i v_{i+1}$ is a lower-left edge. We can assume the first one, otherwise we can take the opposite orientation of S. Let $v_i, \ldots, v_j$ be a maximal interval of S where all edges are lower-right. By assumption, edge $v_j v_{j+1}$ can not be horizontal, by the minimality of S it can not be upper, if it is lower, or lower-left, then we can proceed just like in the case of cycle-separators, by a sequence of edge contractions we can obtain an induced coloring of G_m with two less colors and we are done by induction. So, $v_j v_{j+1}$ can only be an upper-right edge. We can apply the same argument for the next maximal interval $v_j, \ldots, v_k$ and obtain that $v_k v_{k+1}$ is a lower-right edge. We can argue similarly "backwards" on S, if $v_l, \ldots, v_i$ is a maximal interval of upper-right edges, then $v_{l-1}v_l$ is a lower-right edge. It follows, that all edges of S are either upper-right, or lower-right. But then S can not have more then m vertices, a contradiction. In the case when there are no *vertical* edges, the argument is almost exactly the same.

Subcase 2.2. *S starts in $x = 0$, ends in $x = m-1$, and $|S| = m$.*
Since If $|S| = m$, $S = v_1, v_2, \ldots, v_m$ such that $v_i = (i-1, y_i)$, for every i. We show that $G_m \setminus S$ contains a subgraph isomorphic to G_{2k}.

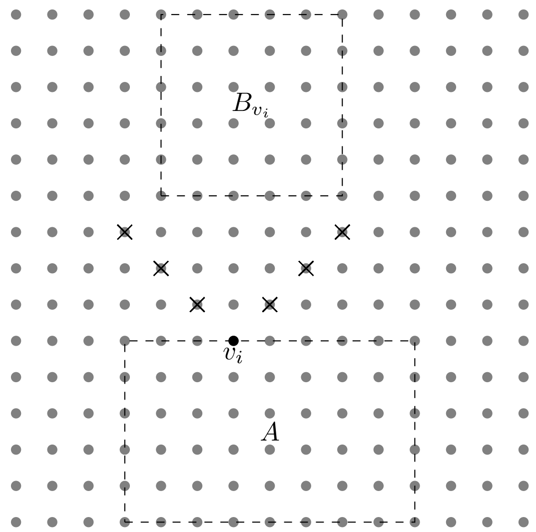

Figure 6: The subcase $|S| = m$

Suppose that $5k \le m \le 5k + 4$. Consider the set of vertices

$$A = \{(x, y) \mid k \le x \le 4k - 1,\ 0 \le y \le 2k - 1\}.$$

Set A induces a $3k \times 2k$ grid graph, $G_{3k,2k}$, in G_m. If $A \cap S = \emptyset$, then $G_m - S \supseteq G_{3k,2k} \supseteq G_{2k}$; otherwise some $v_i \in S$ belongs to A, i.e., $v_i = (i, y_i)$ with $k \le i \le 4k - 1$ and $0 \le y_i \le 2k - 1$. Then, consider the set of vertices

$$B_{v_i} = \{(x, y) \mid i - k + 1 \le x \le i + k,\ 3k \le y \le m - 1\},$$

which contains a G_{2k} subgraph in G_m and it is disjoint from S. Therefore, $G_m - S$ contains a subgraph isomorphic to G_{2k}, and thus $\chi_{\mathrm{um}}(G_m) \ge m + \chi_{\mathrm{um}}(G_{2k}) \ge m + \frac{10}{3}k - 18\log_2 2k \ge \frac{5}{3}m - 18\log_2 m$. □

Remark 5.2. By a slightly more careful calculation we could get $\chi_{\mathrm{um}}(G_m) \ge \frac{5}{3}m - \log_{5/2} m$.

An immediate corollary from theorem 4.8 is the following.

Corollary 5.3. *For $m \ge 2$, $\chi_{\mathrm{cf}}(G_m) \ge \frac{5}{6}m - 10\log_2 m$.*

6 Discussion and open problems

As we mentioned in the introduction, conflict-free and unique-maximum colorings can be defined for hypergraphs. In the literature of conflict-free colorings, hypergraphs that are induced by geometric shapes have been in the focus. It would be interesting to show possible relations of the respective chromatic numbers in this setting.

An interesting open problem is to determine the exact value of the unique-maximum chromatic number for the square grid G_m. In this paper, we improved the lower bound asymptotically to $5m/3$, and we believe that this bound is still far from optimal. Observe that in each case, our recursion step would allow us to prove a lower bound of the form $2m - o(m)$, with the exception of the last case, when $|S| = m$, that is the "bottleneck" of the proof. We believe that using a more complicated recursion, with grids of rectangular shapes, could lead to an improvement.

Another area for improvement is the relation between the two chromatic numbers for general graphs. We have only found graphs which have unique-maximum chromatic number about twice the conflict-free chromatic number, but the only bound we have proved on $\chi_{\mathrm{um}}(G)$ is exponential in $\chi_{\mathrm{cf}}(G)$.

Finally, the coNP-completeness of checking whether a coloring is conflict-free, implies that the decision problem for the conflict-free chromatic number is in complexity class Π_2^p (at the second level of the polynomial hierarchy). An interesting direction for research would be to attempt a proof of Π_2^p completeness for this last decision problem.

References

[1] Amotz Bar-Noy, Panagiotis Cheilaris, Michael Lampis, Valia Mitsou, and Stathis Zachos. Ordered coloring grids and related graphs. In *Proceedings of the 16th International Colloquium on Structural Information and Communication Complexity (SIROCCO)*, 2009.

[2] Amotz Bar-Noy, Panagiotis Cheilaris, Michael Lampis, Valia Mitsou, and Stathis Zachos. Ordered coloring of grids and related graphs. Submitted to a journal, 2009.

[3] Amotz Bar-Noy, Panagiotis Cheilaris, and Shakhar Smorodinsky. Deterministic conflict-free coloring for intervals: from offline to online. *ACM Transactions on Algorithms*, 4(4):44:1–44:18, 2008.

[4] Hans L. Bodlaender, Jitender S. Deogun, Klaus Jansen, Ton Kloks, Dieter Kratsch, Haiko Müller, and Zsolt Tuza. Rankings of graphs. *SIAM Journal on Discrete Mathematics*, 11(1):168–181, 1998.

[5] Hans L. Bodlaender, John R. Gilbert, Hjálmtyr Hafsteinsson, and Ton Kloks. Approximating treewidth, pathwidth, frontsize, and shortest elimination tree. *Journal of Algorithms*, 18(2):238–255, 1995.

[6] Ke Chen, Amos Fiat, Haim Kaplan, Meital Levy, Jiří Matoušek, Elchanan Mossel, János Pach, Micha Sharir, Shakhar Smorodinsky, Uli Wagner, and Emo Welzl. Online conflict-free coloring for intervals. *SIAM Journal on Computing*, 36(5):1342–1359, 2007.

[7] Khaled Elbassioni and Nabil H. Mustafa. Conflict-free colorings of rectangles ranges. In *Proceedings of the 23rd International Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 254–263, 2006.

[8] Guy Even, Zvi Lotker, Dana Ron, and Shakhar Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM Journal on Computing*, 33:94–136, 2003.

[9] Sariel Har-Peled and Shakhar Smorodinsky. Conflict-free coloring of points and simple regions in the plane. *Discrete and Computational Geometry*, 34:47–70, 2005.

[10] Ananth V. Iyer, H. Ronald Ratliff, and Gopalakrishanan Vijayan. Optimal node ranking of trees. *Information Processing Letters*, 28:225–229, 1988.

[11] Meir Katchalski, William McCuaig, and Suzanne Seager. Ordered colourings. *Discrete Mathematics*, 142:141–154, 1995.

[12] Charles E. Leiserson. Area-efficient graph layouts (for VLSI). In *Proceedings of the 21st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 270–281, 1980.

[13] Joseph W.H. Liu. The role of elimination trees in sparse factorization. *SIAM Journal on Matrix Analysis and Applications*, 11(1):134–172, 1990.

[14] Donna Crystal Llewellyn, Craig A. Tovey, and Michael A. Trick. Local optimization on graphs. *Discrete Applied Mathematics*, 23(2):157–178, 1989.

[15] János Pach and Géza Tóth. Conflict free colorings. In *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, pages 665–671. Springer Verlag, 2003.

[16] Christos Papadimitriou. *Computational Complexity.* Addison Wesley, 1993.

[17] Alex Pothen. The complexity of optimal elimination trees. Technical Report CS-88-16, Department of Computer Science, Pennsylvania State University, 1988.

[18] Shakhar Smorodinsky. *Combinatorial Problems in Computational Geometry.* PhD thesis, School of Computer Science, Tel-Aviv University, 2003.